A NEW ADVANCED NON-LINEAR HEAT CONDUCTION MODELLING TECHNOLOGY

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http://www.hao.ucar.edu/public/education/sp/images/newton.html

http://www.answers.com/topic/gottfried-leibniz

Sir Isaac Newton (top) and Gottfried Leibniz (bottom) independently developed differential calculus during the late 17th century. This led to accurate mathematical models of physical reality, which are today widely used in engineering, physics and medicine.

Heat transfer is as obvious to complex mechanical engineering systems as water is to life. The mechanism of heat transfer in solid materials is referred to as heat conduction, and it forms an intrinsic part of systems and machines which range from large scale power generation systems such as nuclear reactors, to thermal machines such as internal combustion engines and aircraft gas turbines. Heat conduction is not only a deciding factor to the energy efficiency of these systems, but also to the safety of critically important components. Its accurate description is therefore vital to the proper and safe operation of existing machines, while being essential to those engineers who take up the challenge of being the architects of tomorrow, by being the innovators of today.

Sir Isaac Newton and Gottfried Leibniz independently laid the foundation to differential calculus, which made the accurate continuum description of physical systems possible. Today, this is referred to as continuum mechanics, and constitutes a theoretical field that allows engineering to be conducted with precision. A prime example is the partial differential equation that employs Fourier's law (which dates back to 1811), to describe multi-dimensional heat conduction in gases, liquids and solids. When employed to describe real complex engineering components, the resulting mathematical expressions, although being accurate, constitute a formidable piece of differential calculus.

Even Newton and Leibniz did not attempt to solve these via analytical means, and no general analytical solution exists to date. To the engineer however, it is the solution to this differential equation that constitutes the critical final step in describing the heat conduction physics in components, as this enables the accurate prediction of quantities such as the temperature field. Newton, as well as the giants of mathematics of the 18th and 19th centuries, made major contributions to the approximate or numerical solution of mathematical problems too complex to solve via analytical means. Foremost among these were Euler (1707-1783), Lagrange (1736-1813) and Gauss (1777-1855). This gave rise to the field known today as numerical analysis, and is currently the method of choice of scientists and engineers who busy themselves with the macroscopic continuum modelling (modelling scales are such that molecular activity is not described explicitly) of complex physical processes and systems.

Numerical analysis involves solving a complex partial differential equation in discrete parts. These discrete parts are often termed discrete domains. In terms of non-linear heat conduction, the discrete domains are mainly spatial i.e. a specific problem is geometrically subdivided into a number of discrete spatial domains. An example of the latter is shown in → 2, where a part of a turbine guide vane → 1 is shown (modelled as a two-dimensional problem) which has been geometrically decomposed into a number of triangular and quadrilateral domains. The governing equations are applied to each discrete domain in such a manner that a set of discrete equations (one equation per vertex) results, which may be solved simultaneously by iterative means.

Even though accepted numerical techniques guarantee only approximate solutions, they have a key characteristic: the error which is made via the approximation which underlies a numerical method, tends identically to zero as the discrete domains tend to zero. The trade-off is that the smaller the discrete domains, the more discrete equations are to be solved (this could be thousands or even millions). As solving even 100 discrete equations simultaneously by hand is an extremely time-consuming task, men such as Newton would not have been able to obtain accurate solutions to the partial differential equation describing the non-linear heat conduction in a realistic engineering component. This is where the advent of the digital computer is of great importance. Problem solved? Not quite...

Even though the latest in digital CPU technology is able to deal with millions of unknowns, the brute force approach does not offer enough memory or solution times that are quick enough for a modern world. There are two conflicting aspects here: required memory and CPU time. Solution algorithms that require low CPU times typically need excessive amounts of memory and vice versa. A technique which is both CPU-time and memory efficient (being matrix-free) therefore has considerable industrial importance, and it is here where the competition is on.
A decidedly complicating aspect to the development of fast efficient solution algorithms, is their applicability to complex spatial decomposition methodologies viz. the use of triangular domains in two dimensions. This is referred to as unstructured decomposition and produces the type of meshes shown in → 3 (triangular domain). The use of unstructured meshes is of considerable importance as this allows natural applicability to the complex geometries prevalent in modern engineering components. Combining structured and unstructured methodologies is however king, as this has impressive computational advantages. This is referred to as unstructured-hybrid spatial decomposition, and the result is shown in the figure.

Efficient hybrid-unstructured solution algorithms which are applicable to large sets of equations may be subdivided into purely algebraic or partly-algebraic-partly-geometric methods. The downside to the latter is that certain complications occur when complex geometries are prevalent (as is the case in engineering). Purely algebraic methods however, elevate the solution process to a new level of abstraction, making them complex to develop and implement. Because of the more generic applicability of the purely algebraic, as well as the preferred use of hybrid-unstructured meshes, it is the method currently under development at the Department of Mechanical and Aeronautical Engineering. The specific algebraic method being researched comes from a family of solution algorithms referred to as Krylov-subspace methods. In the context of non-linear problems, these are also known as generalised minimal residuals (GMRES) methods.

The work conducted by the author has culminated in a GMRES solver which models non-linear heat conduction in complex engineering systems fast and efficiently. A key solver ingredient is Newton-linearisation (as applied to systems of non-linear equations), which is done analytically. The resulting system of discrete equations is then solved via the GMRES algorithm, with the associated Krylov-subspace vectors being preconditioned via lower-upper symmetric Gauss-Seidel (LU-SGS), as first proposed by Luo *et al.*[1] To test the efficiency of the developed technology, the temperature distribution in an experimental aerospace gas-turbine engine guide vane is modelled (a schematic of the vane is shown in → 2). This vane is known as Mark II, and is subjected to gas flows in excess of 780°C.[3] To ensure that the vane material temperatures are kept below certain allowable values, it is provided with internal cooling channels (as shown in the figure). Note that the temperatures predicted with the developed technology compare well with that of Bohn *et al.*[2]

Of particular interest is how fast the developed solver is able to find a solution to the above problem. Shown in → 3 are comparisons of the computational time required by the proposed scheme (LU-SGS+GMRES) with that of other purely algebraic matrix-free methods. It is clear that the former outperforms the other algorithms in terms of required computer time by a factor 20. The premium is that it requires only a third more memory, making it a truly fast and efficient algebraic solver.

References

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Further Reading

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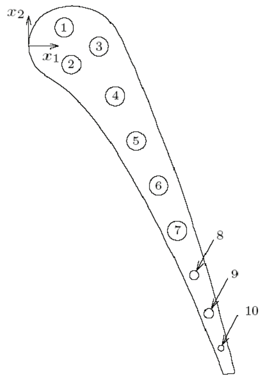
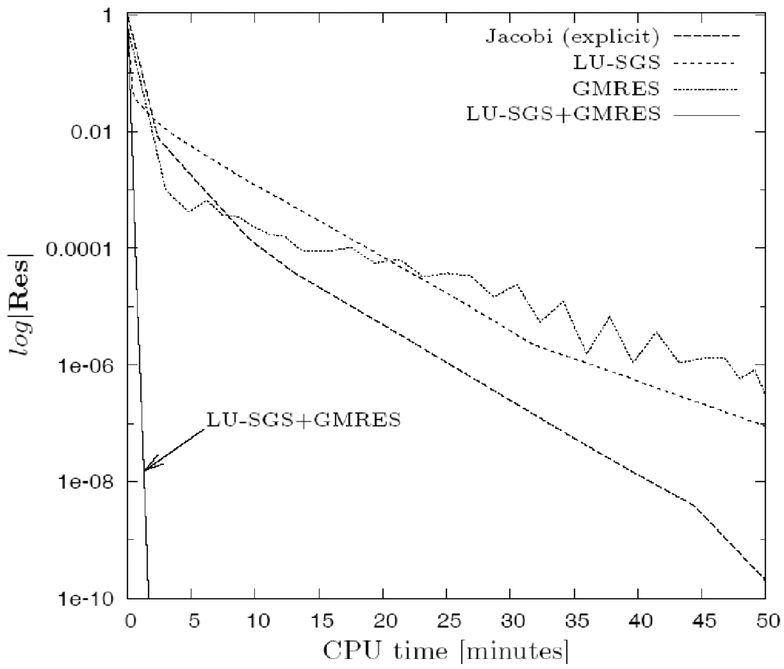


cooling slot nr.	h_c [W/m²K]	T_∞ [K]
1	1943.67	336.39
2	1881.45	326.27
3	1893.49	332.68
4	1960.62	338.86
5	1850.77	318.95
6	1813.36	315.58
7	1871.88	326.26
8	2643.07	359.83
9	1809.89	360.89
10	3056.69	414.85

→ 1. Sectional view of the Mark II gas turbine guide vane. The vane contains 10 cooling slots through which air is circulated for the purpose of ensuring that temperatures remain below safe limits. The effective convective heat transfer coefficient as well as mean temperature of the cooling air in each slot is shown on the table.

→ 2. Close-up view of the hybrid unstructured computational mesh (left) used to model the non-linear heat conduction in a turbine guide vane. The temperatures (in Kelvin) predicted by the developed technology are also shown (right). These compare well with published data.



→ 3. The computational efficiency of the developed solver (LU-SGS+GMRES) is compared to that of other solution algorithms. What is plotted is the residual or error on the vertical axis and actual CPU time on the horizontal axis. The solver clearly outperforms other techniques by a significant margin.